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The Magazine Sponsorship

Despite vigorous protests addressed to L. S. U.'s governing board and to its President—protests that have come from a great many different sections of the United States and from the present editor and manager of the Magazine*—it was decided at the recent meeting of the Board of Supervisors that the recommendation regarding the support of the* Magazine*, which had been made by the University Budget Committee in December, should be approved without change.*

Under this recommendation L. S. U. would contribute only secretarial and administrative aid to the journal, no money being budgeted for printing. Confident that, if these terms of L. S. U. should be accepted, the safety of the journal would be jeopardized we have decided not to accept them. We think it best to seek sponsorship by another institution even as we acknowledge our profoundest gratitude to Louisiana State University for its seven years of service to the Magazine*. Put otherwise, the present editor and manager will after July 1 assume a temporary sponsorship of* National Mathematics Magazine*—this sponsorship to be transferred to the institution or institutions that shall be found willing and able to assume the role presently being performed by L. S. U.*

In the meanwhile, it gives me pleasure to announce to the supporters of the Magazine *that there is in prospect an arrangement which, if it can be effected, will mean that the new sponsorship will be assumed by another Louisiana institution, the writer to be retained as a member of its mathematics staff, and continued as* Magazine *editor.*

The April issue will very probably contain our final announcement on the matter.

S. T. Sanders.

General Comparison of Conformal and Equilong Geometries

By JOHN DE CICCO
Illinois Institute of Technology

The subject of this discussion is a general comparison of the conformal and equilong geometries of the plane. We shall outline the essential features of these two geometries, demonstrating the many similarities and many distinctions. This subject was inaugurated by Scheffers in his fundamental paper of 1905,* and has been continued and completed in a series of contributions by Kasner (1909-1914), and a series of joint papers published recently by Kasner and De Cicco. In many ways, the two theories may be considered to be roughly dual or analogous. However, this is *not* projective duality. There is no general transference principle. Analogous theorems are not consequences of each other as is the case in projective geometry but require separate proofs. The many dissimilar results seem to point to the nonexistence of any exact simple transference principle.

Conformal transformations are the point correspondences of the plane which preserve or reverse the angle between the directions of any two curves at their common point of intersection. On the other hand, equilong transformations (as defined by Scheffers) are the line correspondences of the plane which preserve or reverse the distance between the points of contact of any two curves along their common tangent line. Conformal correspondences are given by functions of the complex variable $x \pm iy$, where $i^2 = -1$; whereas equilong correspondences are given by functions of the *dual* variable $u \mp jv$ where $j^2 = 0$, and where u,v are hessian or better equilong line coordinates.

1. *Curvilinear angles and their duals.* Perhaps the most interesting phase of conformal geometry is the study of the invariant theory of curvilinear angles as developed by Kasner about 1912. Kasner's fundamental theorem states that if the angle $\Theta \neq 0$ is a rational part of π radians, there exists a *unique* absolute differential invariant of higher order (involving the curvature and the different rates of variation of curvature of the sides of the curvilinear angle calculated at the vertex). But if Θ is *not* a rational part of π, no higher invariants exist,

*Scheffers, Equilong geometry, Mathematische Annalen, Vol. 60, pp. 491-531, 1905. This is the paper which inaugurated the subject of equilong geometry.

whether or not a curvilinear angle possesses invariants of infinite order is still unsettled. (These relate to the difficult convergence questions of the power series involved).

In the equilong theory, the corresponding configuration studied is that consisting of two analytic arcs with a common tangent line l such that $S \neq 0$ is the distance between their two points of contact on l. Kasner has proved that S is the *unique* invariant for any two such curves. In other words, there exists an equilong transformation which will carry any pair of such curves into any other pair. Thus the concept of curvilinear angle and its dual produce fundamental distinctions in the conformal and equilong theories.

2. *Horn angles.* A horn angle is an important type of self-dual configuration, which has been studied extensively by Kasner. It consists of an ordered pair of curves which pass through a common point called the vertex, and in a common direction. A horn angle is of order n if the two curve sides have $n+1$ but not $n+2$ consecutive points in common at the vertex that is contact of exactly nth order. The fundamental result is that a horn angle of order n has a *unique* absolute conformal differential invariant of order $2n+1$. This is called the conformal measure of the horn angle. The analogous statement is true in equilong geometry. However, in the conformal case, the question of invariants of infinite order is unsettled, whereas in the equilong theory it has been proved that there are no such invariants.

With all the horn angles of order n through a common point in a common direction, there can be associated an $(n+1)$-dimensional space, the Kasner space K_{n+1}, where the distance formula between any two points is the Kasner metric, namely, the conformal or equilong measure of the corresponding horn angle. We meet here special Finsler spaces, not of riemannian type. The associated group is a $(2n+1)$-parameter group G_{2n+1}. The geometry of the Kasner plane $(n=1)$ and the Kasner three-space $(n=2)$ have been studied by Kasner and the author. In particular, we obtain the trigonometry called trihornometry, of a triangle in both of these two cases. The geometries of the Kasner spaces K_{n+1} are isomorphic for the conformal and equilong theories for $n=1,2$. Recently Mary Elizabeth Ladue has shown that this is also the case for $n=3,4,5$. The reason for this isomorphism is that the algebraic structures for the conformal and equilong horn angle measures are identical for the first five cases. Whether this is true for all cases is unknown.

3. *Elements of third order.* The conformal group operating upon the ∞^3 differential elements of third order passing through a fixed

point induces the six-parameter Kasner-Comenetz group. By making the inclination constant, there results the three-parameter group G_3 which yields the geometry of the Kasner plane. The geometry of our larger group contains many results of G_3 as special cases. Three third order elements possess a new fundamental invariant which is essentially a generalization of Kasner's conformal measure of a horn angle of first order. Many other results about this group are obtained which are too numerous to mention here.

In the dual case, we obtain an analogous six-parameter group which, however, is *not* isomorphic to the Kasner-Comenetz group. The algebraic structure of the fundamental invariant which is a generalization of Kasner's equilong measure of a horn angle of first order is entirely different from that in the conformal theory.

4. *Families of Curves.* Any two isothermal families of curves are conformally equivalent. We may mention the following from the many characteristic properties of isothermal families. The associated point correspondence is conformal. The conjugate, the isocline, and any isogonal families are isothermal systems. Analogous results are valid for dual-isothermal systems.

Any simple non-isothermal family (∞^1 curves) possesses two fundamental conformal covariants of the third order. Moreover if any two families possess the same two covariants, there exist $2\infty^2$ conformal transformations which will convert one into the other. These covariants characterize the conformal group. (A different treatment of the conformal theory of families is being published in Bull. Am. Math. Soc. by Robert Coleman). Similar statements are true in the dual case.

Here we briefly mention certain systems of ∞^2 and ∞^3 curves which have certain interesting geometric properties and which may serve to characterize conformal and equilong groups. For the conformal theory, these are natural families, velocity families, Ω families; and for the equilong theory, their duals.

5. *Transformations of period two.* Kasner has proved that the set of all conformal transformations of period two consists of *two* distinct types: (T_1) conformal involutions and (T_2) conformal symmetries (Schwarzian reflections). Any conformal symmetry is equivalent to Mœbius inversion.

In the equilong theory, it is found that there are *three* distinct types of equilong transformations of period two: (T_1) equilong involutions, (T_2) K symmetry, and (T_3) D inversion. K symmetry is equiv-

alent to K inversion in a circle; whereas D inversion is equivalent to Laguerre inversion.

We thus have five types of transformations of period two: two conformal and three equilong.

Any K symmetry may be regarded as the dual of conformal symmetry. It is the unique reverse equilong transformation which keeps fixed the tangent lines of a given curve C. When C is known, the construction of conformal symmetry is extremely difficult; whereas the construction of K symmetry is quite easily accomplished by reflecting each line l of the plane through the tangent line t on C parallel to l.

7. *Infinite groups.* It is a fundamental theorem of Kasner that the group generated by all conformal transformations of period two is identical with the group generated by all conformal symmetries. Any conformal transformation can be factored into the product of some transformation of this group followed by a magnification.

The analogous result in the equilong theory is quite dissimilar. The group generated by all equilong transformations of period two is identical with the group generated by all K symmetries and D inversions. But this group is much smaller than the total equilong group, whereas in the conformal theory the corresponding group is almost the complete conformal group.

8. *Conclusion.* Other topics discussed whose ideas have been dualized are Mœbius and Laguerre groups, near-Mœbius and near-Laguerre transformations, isogonal and equitangential series, and finally whirl series, which are self-dual configurations.

The preceding is but a brief discussion of some of the outstanding results. These two theories stand out both for their many similarities and many striking dissimilarities.

BIBLIOGRAPHY

See the following papers by Kasner on conformal and equilong geometries:

1 Conformal geometry, Proceedings of the Fifth International Congress of Mathematicians, Vol. 2, p. 81, 1912.

2. Infinite groups generated by conformal transformations of period two (Involutions and symmetries), American Journal of Mathematics, Vol. 38, pp. 177-184, April, 1916.

3. Equilong invariants and convergence proofs, Bulletin of the American Mathematical Society, Vol. 23, pp. 341-347, May, 1917.

4. Trihornometry: A new chapter of conformal geometry, Proceedings of the National Academy of Sciences, Vol. 23, pp. 337-341, June, 1937.

5. The geometry of isogonal and equitangential series, Transactions of the American Mathematical Society, Vol. 42, pp. 94-106, July, 1937.

6. Geometry of conformal symmetry (Schwarzian reflection), Annals of Mathematics Vol. 38, pp. 873-879, October, 1937.

7. Families of curves conformally equivalent to circles, Transactions of the American Mathematical Society, Vol. 49, pp. 378-391, May, 1941.

8. Lineal element transformations which preserve the isothermal character, Proceedings of the National Academy of Sciences, pp. 406-412, August, 1941.

9. Conformal geometry of third order differential elements, Rivisti di Mathematica (Argentina), 1942.

10. Infinite groups generated by equilong transformations of period two, American Journal of Mathematics, Vol. 63, pp. 709-725, October, 1941.

An Important New Text for College Classes

Intermediate Algebra for College Students

By T. S. PETERSON
University of Oregon

This is an unusually attractive textbook in intermediate algebra, and has the advantage of having been carefully tested in lithoprinted form. Among its important features are an exceptionally thorough presentation of elementary algebra, many illustrative examples, an abundance of carefully graded problems, adaptability to students of varying degrees of preparation, provision for periodic reviews. Answers to odd problems are included in the text.

HARPER & BROTHERS, 49 East 33d St., New York

Primitive Integral Triangles

By H. T. R. AUDE
Colgate University

It is known that a one-to-one correspondence exists between all pairs of relatively prime integers $m,n(m>n$, not both odd) and all primitive integral right triangles whose sides are represented by the triads m^2-n^2, $2mn$, m^2+n^2. This relationship may be taken as based upon the geometric fact or property that the tangents of half of the acute angles of the right triangles are n/m and $(m-n)/(m+n)$, respectively.

This paper begins by setting up a similar relationship for triangles that have integral sides and an angle of 120°. All relatively prime number pairs are separated into two groups with their elements in one-to-one correspondence. Thereupon the elements of either group are shown to be in one-to-one correspondence with all the primitive integral 120° triangles. From this there evolves by a geometric relationship all the primitive integral triangles with one angle of 60°. Finally, these methods are applied to any triangle with integral sides where the cosine of one angle is specified, showing that the triads which represent the sides of such triangles can be obtained for a specified range of, say, the perimeter without repetition. Furthermore, since for every primitive integral triangle there corresponds a set of three rational cosines and conversely, it follows that all primitive integral triangles come within the scope of this paper.

In the triangle ABC, where the respective opposite sides are represented by the integers a, b, and c, the angle C is assumed to be 120°. The law of sines with other formulas lead to the relationship

$$(1) \qquad a/(b+c)=\sin A/(\sin B+\sin C)$$
$$=\cos \tfrac{1}{2}(102°+B)/\cos \tfrac{1}{2}(120°-B)=n/m$$

where m and n are integers relatively prime, $m>n$. Solving (1) for c gives

$$(2) \qquad c=am/n-b.$$

This value of c substituted in the law of cosines $c^2=a^2+b^2+ab$ gives an equation, which solved for a/b, yields the value

$$(3) \qquad a/b=(2mn+n^2)/(m^2-n^2).$$

Only the primitive, or non-similar, 120° triangles need to be considered. Whence, it is preferable that the three integers a, b, and c be taken relatively prime. It will be shown later that a choice of the number pair m,n can be made so that the numbers $2mn+n^2$ and m^2-n^2 in relation (3) are relatively prime. Therefore, take these numbers, respectively, as the values for a and b; the number c is found by (2) and we have

$$a=2mn+n^2, \quad b=m^2-n^2, \quad c=m^2+n^2+mn. \tag{4}$$

Whence for every number pair m,n there exists a 120° triangle whose sides are represented by the triad a, b, c as given in (4).

At this point it is well to observe that in place of (1) we could have written

$$b/(a+c)=\sin B/(\sin A+\sin C)=n'/m'.$$

where the number pair m',n' will play a role similar to that of the number pair m,n. By using the values a, b, and c as given in (4) it is found that the value of the ratio n'/m' is

$$n'/m'=(m^2-n^2)/(m^2+3mn+2n^2)=(m-n)/(m+2n).$$

From this relation it appears that if a certain number pair m,n will produce a certain triad a, b, c; then the same triad, or one proportional thereto, should come forth by using the number pair $m+2n,m-n$. This is indeed the case. For the number pair m',n' or $m+2n,m-n$ when used in (4) gives the triad

$$a=3(m^2-n^2), \quad b=3(2mn+n^2), \quad c=3(m^2+n^2+mn).$$

This set represents a triangle similar to the one given by (4) with an interchange of the sides adjacent to the 120 degree angle. There are now two number pairs m,n and $m+2n,m-n$ which lead to, and hence may be said to determine, a certain primitive 120° triangle with the sides a, b, c.

That these two number pairs are the only ones will become evident by observing that there is a one-to-one correspondence between all relatively prime number pairs and all the proper rational fractions when in the lowest terms, and that all such fractions can be separated into two groups I and II, where for every fraction n/m of group I, there exists a fraction $n'/m'=(m-n)/(m+2n)$ in group II, and conversely, since $n/m=(m'-n')/(m'+2n')$. When the difference, denominator minus numerator, for a fraction is a multiple of 3 that fraction will belong to group II, otherwise it will be of group I. Finally, with this grouping in effect, since no fraction of group I equals any fraction

of group II, and since all the fractions of type $(m-n)/(m+2n)$ belong to group II and are in lowest terms, there exists a one-to-one-to-one correspondence between the relatively prime number pairs m,n and $m+2n,m-n$ and the triads a, b, c which may represent the sides of 120° triangles.

Therefore, to obtain all the primitive integral 120° triangles for which, say, the side opposite the 120° angle is less than a certain number p, write all the relatively prime number pairs m,n $(m>n)$ such that m^2+n^2+mn is less than p. Strike out all those pairs of numbers for which $m-n\equiv 0$, modulo 3. The remaining number pairs will produce by (4), once and once only the 120° triangles for the specified range.

The choice of the two numbers which were used in the discussion leading up to the triad given in (4) can now be justified. For the condition $m-n\equiv 0$, modulo 3, implies the two congruences $2mn+n^2\equiv 0$, $m^2-n^2\equiv 0$, modulo 3. Furthermore the elimination of the number pairs m,n where $m-n$ has 3 as a factor leaves the two numbers $2mn+n^2$ and m^2-n^2 relatively prime.

By a geometric relationship it is easy to pass from the 120° integral triangles to the integral triangles that have but one angle 60°. For each triangle of the first kind there exists two of the second. For the first, add on the exterior of any 120° triangles with sides a, b, c the equilateral triangle constructed on the side a. For the second, add on the equilateral triangle built on the side b. There will result the two distinct 60° triangles with the sides a, $a-b$, c and $a-b$, b, c, respectively. The converse holds, so that this relationship will give all primitive integral triangles having but one 60 degree angle. Thus for every relatively prime number pair m,n $(m>n)$ and $m-n\equiv 0$, modulo 3, there exists the two primitive 60° triangles whose sides are represented, respectively by the two triads

(5) $2mn+n^2$, $2mn+m^2$, m^2+n^2+mn; $2mn+m^2$, m^2-n^2, m^2+n^2+mn.

Next, consider the general case where it is assumed that the cosine of the angle C is equal to r/s, a ratio of two relatively prime integers of which r may be negative.

Upon eliminating c between the equation

$$a/(b+c)=n/m \tag{6}$$

and the law of cosines it will be found that the ratio a/b has the value

$$a/b=(2mns-2n^2r)/(m^2s-n^2s). \tag{7}$$

Consequently the three numbers a, b, c, which represent the sides of the desired triangle, are given by the proportion

$$a:b:c=2n(ms-nr):(m^2-n^2)s:(m^2s-2mnr+n^2s), \tag{8}$$

written in this form so that any common factor can be removed.

But it would have served equally well in relation (6) to have written

$$b/(a+c)=n'/m'. \tag{9}$$

The result would be similar, but with the numbers m and n replaced respectively by the numbers m' and n', and an interchange of the numbers a and b. Using the relations in (8) it is found that the proper fraction n'/m' in (9) has the value

$$n'/m'=(m-n)s/(sm+sn-2nr). \tag{10}$$

It is thus seen that number pair m',n' as defined in (10) will yield essentially the same triangle as the number pair m,n.

If a pair of numbers m,n by (8) produce a triad a, b, c which represents an integral triangle with $\cos C=r/s$, then the pair of numbers $(m+n)s-2nr$, $(m-n)s$ will produce a triad a', b', c' such that

$$a' : b=b' : a=c' : c=2s(s-r).$$

The triangle represented by the numbers a', b', c' is essentially the same triangle.

This paper will now be concluded by solving the problem: Find all the integral triangles which have the cosine of one angle equal to $-1/4$ and a perimeter less than 100 units.

With $r=-1$ and $s=4$, it is found that the sides are

$$a : b : c=(4mn+n^2) : 2(m^2-n^2) : (2m^2+mn+2n^2).$$

Proceeding systematically, beginning with the smallest number pair for m,n and remembering that the pair $2m+3n$, $2m-2n$ will give essentially the same triangle as the pair m,n, the desired triads a, b, c are obtained. As the work evolves it is seen, and it can easily be proved, that a common factor to a, b, and c will exist whenever n is an even number, and when $m+n\equiv 0$ modulo 3. The number pairs which will produce the required triads are: 2,1; 3,1; 3,2; 4,1; 5,1; 5,2; 5,4; and 8,1. The triads for the primitive triangles are, respectively: 3,2,4; 13,16,23; 14,5,16; 17,30,38; 7,16,19; 22,21,34; 16,3,17; 11,42,46; giving eight primitive integral triangles which have the cosine of an angle equal to $-1/4$ and a perimeter less than 100 units.

Humanism and History of Mathematics

Edited by
G. WALDO DUNNINGTON

A History of American Mathematical Journals

By BENJAMIN F. FINKEL
Drury College

(Continued from November, 1941, issue)

Following the *American Journal of Mathematics,* in chronological order of publication, was the *Mathematical Visitor,* edited and published by Artemas Martin, at Erie, Pa., the first number of volume I bearing the date of October, 1878.

The first volume of the *Visitor* contains six numbers, and a total of 198 4-to pages, $7\frac{1}{4}$ by $10\frac{1}{2}$ inches. No. 2 bears the date of Jan., 1878, and contains 38 pp.; No. 3 bears the date of Jan., 1879, and contains 36 pp.; No. 4 bears the date of Jan., 1880, and contains 36 pp.; No. 5 bears the date of July, 1880, and contains 40 pp., and No. 6. bears the date of Jan., 1881, and contains 38 pp.

THE
MATHEMATICAL
VISITOR

Edited and Published by
ARTEMAS MARTIN, M. A.,
Member of the London Mathematical Society

VOLUME I.
1877-1881.

ERIE, PA.
1881.

List of Contributors, III-IV.

The list of contributors consists of 139 mathematicians, mostly American. We shall give the names of a score or more of the most prominent as follows: Edward E. Bowser, Professor of Mathematics, Rutgers College, New Brunswick, N. J.; Dr. Edward Brooks, later superintendent of schools, Philadelphia, Penn.; W. W. Beman, Assistant Professor, later Professor of Mathematics, University of Michigan; Marcus Baker, U. S. Coast and Geodetic Survey Office, Washington, D. C.; Dr. Joseph Ficklin, Professor of Mathematics and Astronomy, Missouri State University, Columbia, Missouri; Dr. Joel E. Hendricks, Editor and Publisher of the *Analyst*, Des Moines, Iowa; Henry Heaton, Perry, Iowa; William Hoover, later Professor of Mathematics, Ohio University, Athens, Ohio; Dr. David S. Hart, Stonington, Conn.; William Woolsey Johnson, Professor of Mathematics, St. John's College, Annapolis, Md.; Dr. Daniel Kirkwood, Professor of Mathematics, University of Indiana, Bloomington, Ind.; Charles H. Kummell, U. S. Coast and Geodetic Survey Office, Washington, D. C.; Miss Christine Ladd, Fellow of Johns Hopkins University, later Professor of Natural Science and Chemistry, Howland School, Union Springs, N. Y.; Dr. Alexander Macfarlane, Edinburg, Scotland; F. P. Matz, Professor of Mathematics, Military and Scientific School, Kings, Mountain, N. C.; Benjamin Peirce, Professor of Mathematics, Harvard University; J. J. Sylvester, LL. D., Professor of Mathematics, Johns Hopkins University; E. B. Seitz, Professor of Mathematics, North Missouri State Normal School, Kirksville, Mo.; Professor Ormond Stone, Astronomer at the Cincinnati Observatory; Walter Siverly, Oil City, Penn.; De Volson Wood, Professor of Mathematics and Mechanics, Stephens Institute of Technology, Hoboken, N. J.

Contents of Volume I.

Number 1. Introduction, p. 1; Junior Problems, p. 2; Senior Problems, p. 8; Solution of a Problem in Probabilities, p. 9; The Intrinsic Equation of a Curve, p. 10; Notices of Periodicals, p. 12.

Number 2. Solutions of Junior Problems Proposed in No. 1, p. 13; Solutions of Senior Problems Proposed in No. 1, p. 23; Solutions of "Unsolved Problems" Published in No. 1, p. 39; List of Contributors, p. 42; Junior Problems, p. 42; Senior Problems, p. 44; Unsolved Problems, p. 48; Editorial Notes, p. 49; Notices of Books and Periodicals, p. 49; Corrigenda, p. 50.

Number 3. Junior Department—Solutions of Problems Proposed in No. 2, p. 51; List of Contributors to the Junior Department, p. 60; Problems, pp. 61-62; Senior Department—Solutions of Problems Pro-

posed in No. 2, p. 62; List of Contributors to the Senior Department, p. 82; Problems, p. 83; Editorial Notes, p. 85; Notices of Books and Periodicals, p. 85; Corrigenda, p. 86.

Number 4. Junior Department—Solutions of Problems Proposed in No. 3, p. 87; List of Contributors to the Junior Department, p. 96; Problems, p. 97; Senior Department—Solutions of Problems Proposed in No. 3, p. 99; List of Contributors to the Senior Department, p. 114; Problems, p. 114; Editorial Notes, p. 118; Notices of Books and Periodicals, p. 119; Corrigenda, p. 120.

Number 5. Junior Department—Solutions of Problems Proposed in Nos. 1, 2, and 3, p. 121; Note on the Solution of Problem 110, p. 124; Problems, p. 125; Senior Department—Solutions of Problems Proposed in No. 3, p. 126; Solution of "Unsolved" Problem 40, p. 147; On Evolutes and Involutes, p. 148; Computation of Sturm's Functions, p. 151; Expansion of Polynomials, p. 151; The Day of the Week for a Given Date, p. 155; Demonstration of two Diophantine Theorems, p. 156; Solution of the "Three Dice" Problem, p. 157; Problems, p. 158; Editorial Notes, p. 159; Notices of Books and Periodicals, p. 159; Corrigenda, p. 160.

Number 6. Junior Department—Solutions of Problems Proposed in No. 4, p. 161; List of Contributors to the Junior Department, p. 175; Problems, p. 175; Senior Department—Solutions of Problems Proposed in No. 4, p. 177; List of Contributors to the Senior Department, p. 192; Problems, p. 193; Editorial Notes, p. 195; Notices of Books and Periodicals, p. 196; Corrigenda, p. 198.

Most of the problems in *The Visitor* are quite difficult. The "Three Dice" problem was proposed and solved by E. B. Seitz. It is stated as follows: If three dice be piled up at random on a horizontal plane, what is the probability that the pile will not fall down? The solution covers page 157 and the answer is:

$$p=\frac{\pi}{(\pi+4)^2}\left[\frac{\pi}{4}+\frac{1}{96}-\frac{\sqrt{2}}{12}+\frac{1}{12}\log(\sqrt{2}-1)\right]$$

A similar problem proposed by the Editor, *Artemas Martin*, and solved by him on pages 9 and 10 is the following: If three pennies be piled up at random on a horizontal plane, what is the probability that the pile will not fall down?

The answer to this problem is

$$p=\frac{1}{16}-\frac{3}{16\pi}\left(\frac{3}{16}\sqrt{15}-2\sin^{-1}\tfrac{1}{4}\right).$$

The problems in this volume being proposed by some of the best mathematicians in the country are of a very high order of excellence.

Many of the problems were proposed by the Editor. Then we find such contributors of problems as Professor Benjamin Peirce, Professor H. A. Wood, Dr. Daniel Kirkwood, Miss Christine Ladd, J. J. Sylvester, Dr. Joel E. Hendricks, Dr. Joseph Ficklin, and Dr. S. Hart Wright.

THE MATHEMATICAL VISITOR

Volume 1 October, 1878 Number 1

INTRODUCTION

In England and Europe periodical publications have contributed much to the diffusion of mathematical learning, and some of the greatest scientific characters of those countries commenced their mathematical career by solving the problems proposed in such works.

It was stated nearly three-quarters of a century ago that the learned Dr. Hutton declared that the *Ladies' Diary* had produced more mathematicians in England than all the mathematical authors in that Kingdom.

Similar publications have produced like results in this country. Not a few of our ablest teachers and mathematicians were first inspired with a love of mathematical science by the problems and solutions published in the mathematical department of some unpretending periodical.

Turning over the leaves of the *Mathematical Miscellany*, published about forty years ago, we find there the names of Professors Peirce, Strong, Kirkwood, Root, Docharty, and many others, who have distinguished themselves in various departments of mathematics.

Some years ago we asked a young mathematician of great promise, who displayed great ingenuity and ability in solving difficult probability problems, what works he had treating on that subject; imagine our surprise when he replied that he had no works on probabilities, but, had learned what he knew about that difficult branch of mathematical science by studying the problems and solutions in the *Schoolday Magazine*. The young man referred to was E. B. Seitz. F.

Believing that there is room for one more mathematical periodical, we have ventured to issue the *Mathematical Visitor*, which will be devoted chiefly to problems and solutions; and we invite professors, teachers, students, and all lovers of the "bewitching science," to contribute their best problems and solutions for its pages.

Two lists of problems will be published in each number; one, headed "Junior Problems", for students and persons who have not advanced very far beyond the elementary branches; the other, headed "Senior Problems", will contain questions of a more difficult nature. For convenience of reference the problems will be numbered continuously.

A prize problem will be published in each number, for the best complete, correct, solution of which ten copies of *The Visitor* will be given, and for the second best solution eight copies will be given.

All problems and solutions intended for publication in number two should be received by September 1, 1877. Number 2 will be issued about January 1, 1878; it will contain about 32 pages and the price will be 50 cents.

Only a small edition will be printed, hence persons desiring to secure copies should send in their names at an early date.

ARTEMAS MARTIN, Lock Box 11, Erie, Pa.

The type setting and printing of *The Mathematical Visitor* was done by the Editor, he having bought for the purpose the necessary type and a hand printing press. Just a short time before his death he offered to give to the *Writer* of this history, his entire equipment.

The mechanical execution of *The Visitor* was considered by competent judges to be the finest mathematical typework done anywhere in the world. The wood-cuts for the diagrams accompanying Geometrical solutions of problems are unusually fine.

In reference to the type-setting, the Editor says in Volume 1, Number 4, January, 1880, page 118 of *The Visitor*, "This number of *The Visitor* has been delayed some months in consequence of the sickness of the Editor, who has done all the type-setting with his own hands. He is not a practical printer, and never had set up a stickful of type till last May or June."

"The cover, the first six pages, and page 96 were printed by the Editor on a No. 3 Self-Inker Model Press; pages 93, 94, and 95 were printed at the Erie Morning Dispatch office, and the remaining pages at the Erie Gazette office."

"After many delays on account of sickness of the Editor, No. 1 of Vol. II, containing 30 4to pages was announced in the April, 1882, No. of the *Mathematical Magazine*, edited by Dr. Martin. Also after many delays, No. 2 of Vol. II, containing 34 4to pages was announced in the Oct., 1884, No. of the *Mathematical Magazine* as published; No. 3 was announced in the April, 1890, No., containing 38 4to pages and No. 4 of Vol. II, was announced as recently issued in Vol. II, Jan., 1895, No. 9 of the *Mathematical Magazine*.

No. 4 of the *Mathematical Visitor* contains solutions of a number of interesting and difficult "Average" and "Probability" problems by Henry Heaton, the late Professor Seitz, and Professor Zerr; also solutions of the "Duck" Prize Problem, by Professor Bowser, Henry Heaton, the late Dr. Hendricks, and the Editor. It is believed that No. 4 of Vol. II was the last No. of the *Visitor* to be published."

From these statements one may well understand the reason for the iregularity of the appearance of both of Dr. Martin's publications, *The Mathematical Visitor* and *The Mathematical Magazine.*

It was with a view of having a purely mathematical journal reach its readers more regularly at definite stated times, that the *Writer* of this *History* established *The American Mathematical Monthly* at Kidder, Missouri, in January, 1894. F.

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The Teacher's Department

Edited by
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Interrelations Between Mathematics and Philosophy in the Last Three Centuries*

By E. S. KENNEDY
Texas College of Arts and Industries

1. *Introduction.* The present paper is an attempt to characterize the mathematics and the philosophy of each of the last three centuries, with an eye to pointing out influences of each one of the two fields on the other. It will be seen that: (1) The predominant rationalistic philosophy of the seventeenth century was largely the result of successful applications of mathematics to the physical sciences. (2) The eighteenth century empiricists generally were uninfluenced by current mathematics, nor did they influence it. (3) While the nineteenth century found the situation in both fields vastly more complicated, certain common trends can be pointed out.

It is realized that any such broad generalizations as the above may be over-simplified; the hundred-year interval utilized is quite arbitrary, and one is doubtful as to whether or not the whole discussion proves anything. Nevertheless, the exposition which follows may be of some intrinsic interest.

2. *The Seventeenth Century.* It has often been observed that while mathematics as such is purely abstract, having nothing to do with the world of things, nevertheless, the eras of greatest progress in pure mathematics have always accompanied periods when interest in the humanities was high, and when men were bent upon solving and interpreting the practical facts of nature. There is a school of thought which holds that this relation is causal, that the thread of mathematical

*Presented to the Southeastern Section of the Mathematical Association of America March 28 1941.

research is so intertwined with the warp and woof of everyday human experience that it cannot exist alone. So soon as mathematics loses the common touch, say the proponents of this theory,* so soon as mathematicians constitute themselves the high priesthood of an exclusive cult, just so soon will the study itself languish and die. Now, as to whether this relation between the abstract and the mundane is indeed that of cause and effect is a question, but that the correspondence itself does exist cannot be denied.

Never in history has this been better exemplified than during the seventeenth century of our era. It was an age of observation, and the mathematicians found ready to their hands neat bundles of postulate systems—self-evident truths, they naively call them—the very raw material of which mathematics is made. This was by no means the first time that men had sought to deduce bodies of logical doctrine from sets of "clear and distinct ideas". Plato, many centuries before, had defined the province and content of mathematics† in terms that hardly have been improved upon to this day; but Plato also, in encouraging the study of theoretical astronomy,‡ had cautioned his Academicians that it were better for their astronomy if they never looked at the stars. Herein lay the strength—and the fatal weakness—of the austere Greek mathematics. The thinkers of the seventeenth century, on the other hand, dealt with axioms no less abstract, but axioms which were drawn from direct observation of natural phenomena. Fortunately for them the fields they chose to study, mechanics for the most part, were readily susceptible to mathematical investigation. Newton's laws of motion, for instance, formed a system sufficiently simple for abstraction yet sufficiently accurate also in its explanation of the movements of things. So the new mathematics really worked. Galileo's marbles rolled down the board just as he had said they would roll; Kepler's planets appeared in the precise spot in the heavens where he had predicted their appearance, and everyone was enchanted with the success of it all. If it works in mechanics, said they, why will it not work equally well in all physics, in biology, in the social sciences? So Grotius wrote on international law—in the style of Euclid; Spinoza wrote ethics with a geometrical precision, and a Q. E. D. after each theorem. People became imbued with a boundless faith in the methods of mathematics. Laplace,§ in typically modest fash-

*Cf., for example, Hogben, "Mathematics for the Million". In this connection, the author acknowledges the many fruitful suggestions of Dr. T. T. Lafferty of Lehigh University. Dr. W. R. Transue, also of Lehigh, by kindly consenting to read the proofs, saved them the perhaps hazardous trip to the Middle East and back.

†The Republic VI, p. 510.

‡*Ibid.*, VII, p. 529.

§Laplace was seventeenth century in outlook, though he lived in the eighteenth.

ion, announced himself as ready and willing to reconstruct all the history of the past and foretell the future by means of these methods. In his "Théorie Analytique des Probabilités"* he says: "Une intelligence qui pour un instant donné, connaîtrait toutes les forces dont la nature est animée, ... , si d'ailleurs, elle etait assez vaste pour soumettre ces données á l'analyse embrasserait dans le même formule les mouvements des plus grands corps de l'univers et ceux du plus leger atome; rien ne serait incertain pour elle, et l'avenir comme le passé serait present á ses yeux."

The case of the English philosopher Hobbes is also typical. Quoting from Robertson,† who quotes in turn from Aubrey: "He was forty years old before he looked on geometry, which happened accidentally; being in a gentleman's library, Euclid's Elements lay open, and it was the forty-seventh proposition, Book I. So he reads the proposition. 'By Gad,' says he, 'this is impossible!' So he reads the demonstration which referred him back to another, *et sic deinceps*, that at last he was demonstratively convinced of the truth." Regardless of how the irascible Hobbes may have entered the gentleman's library that day, he left it a devoted convert to geometry and its methods. So great was his enthusiam, in fact, that he became a geometer himself, and joined the ranks of the great army of circle-squarers, deluded mortals who were convinced they had solved a problem since proved impossible of solution. For twenty years Hobbes battled doughtily with the great Wallis, the latchet of whose mathematical shoes he was not worthy to unloose, and he was always worsted, but never once did he lose faith in the universal validity of logical deductions.

Thus the path of the seventeenth century mathematician was rosy enough, and it led straight toward rationalism. But so soon as the mathematician turned philosopher, and many of them did, witness Leibniz and Descartes, things became more difficult. The situation was this: abstractions are mind-stuff, matter is obviously not. What then is the connection between the two? The answers were many and varied. Leibniz set up his ingenious system of monads, which according to E. T. Bell,‡ explained absolutely everything in the universe—except the monads. In general, the rationalists held that the two, "on the one hand Matter, with its single location in space and time, on the other hand Mind, perceiving, suffering, reasoning, but not interfering",§ were mutually independent, and God would not let Matter fool Mind, because God was a rationalist Himself. Many of

*Introduction p. ii.
†"Hobbes", p. 31.
‡Men of Mathematics", p. 128.
§Whitehead—"Science and the Modern World" p. 83.

the mathematicians never got far enough along in their rationalism to see the philosophical necessity of God to keep Matter in hand. The story* runs that when Laplace presented Napoleon with a copy of his monumental "Méchanique Céleste", the emperor reproached the savant with having written so large a book about the universe and never once having made mention of its Creator. "Sire", replied Laplace baldly, "je n'avais pas besoin de cette hypothése-la." It would seem indeed that Laplace had need of that "hypothése", but he never realized it.

In first attempting to verify his conjectured law of universal gravitation by means of the moon, Sir Isaac Newton used a slightly erroneous value for the length of a degree at the earth's equator.† The result seemed to indicate a lunar system not quite in equilibrium, a system in which there was need of a divine Hand intervening from time to time, to keep the celestial ball rolling, as it were. The announcement of a more correct value for the offending constant enabled Sir Isaac also to dispense with his "hypothése".

Such, then, was the metaphysics of the seventeenth century, "a scheme of scientific thought framed by mathematicians for the use of mathematicians."‡ For an evaluation we turn again to Whitehead.§ "The enormous success of the scientific abstractions.... has foisted onto philosophy the task of accepting them as the most concrete rendering of fact. Thereby modern philosophy has been ruined. There are the dualists, who accept matter and mind as on equal basis, and the two varieties of monists..... But this juggling with abstractions can never overcome the inherent confusion introduced by the ascription of misplaced concreteness to the scientific scheme of the seventeenth century."

3. *The Eighteenth Century.* In studying the interrelations between mathematics and philosophy during modern times, it will be found illuminating to make a plot against time of the life-spans of the mathematicians involved. When this is done, a curious fact is brought to light. It will be seen that while both the seventeenth and nineteenth centuries are well filled, there is a definite lacuna extending through the greater part of the eighteenth century. Newton and Leibniz, independent inventors of the calculus, had passed from the stage by 1727 and 1716 respectively, while Gauss, the "prince of mathematicians", was not born until 1777, and Cauchy, the apostle of the new rigor, in 1789. It is not to be supposed that there were no great mathe-

*Cajori "History of Mathematics", p. 262, and others.
†*Ibid.*, p. 199.
‡Whitehead—"Science and the Modern World", p. 81.
§*Ibid.*, p. 82.

maticians during the century. On the contrary, the trinity of analysts, Lagrange, Laplace, and Euler, formed a constellation of genius of the first magnitude, and there were lesser lights too numerous to mention. But the significant fact is that none of the last named made contributions which revolutionized the mathematical outlook of their day; their work lay in development rather than in innovation. Thus Lagrange in his "Méchanique Analytique" "did for general mechanics what Newton's law of universal gravitation had done for celestial mechanics";* Laplace's great work was the proof of the stability of the solar system; and Euler, who "calculated without apparent effort, as men breathe, or as eagles sustain themselves in the wind, systematized and unified vast tracts cluttered with partial results and isolated theorems, clearing up the valuable things by the easy power of his analytical machinery."†

Reasons for this trend are not difficult to discover. It will be remembered that in the seventeenth century philosphy and mathematics were almost synonymous—the greatest philosophers were often the greatest mathematicians. But the wave of empiricistic reaction against the "innate truths" of the rationalists washed out a gulf between the two studies, and they remained sundered until the next century. Sated with overmuch deduction, the philosophers turned in relief to studies in which the human individual was more important—psychology, and the like. As for the mathematicians, they were glad to be left in peace to develop the amazingly powerful weapon given them in the new calculus—a weapon so powerful that by its aid any novice was enabled to solve almost automatically problems that had taxed the genius of an Archimedes. Newton has likened himself to a boy playing with shells and pebbles at the seashore, whilst the great ocean of truth lay all undiscovered at his feet, and it may be that the metaphor can be extended to his successors. They too were playing on the beach, oblivious of the jeers of their erstwhile playmates, the philosophers, and so absorbed in building castles of the pebbles Newton had found for them that they had no time to notice the unstable sands on which they built their structures. The result was a phenomenal growth in mathematical knowledge, a growth that was not altogether healthy because of a lack of critical appraisal of results and methods, either from within or from without. It was d'Alembert, living during this period, who coined the maxim: "Venez en avant, et la foi vous viendra", and many of the mathematical indiscretions committed by his contemporaries would seem to call for the exercise of a great deal of faith. Knopp, in his "Theorie und An-

*Bell, "Men of Mathematics", p. 154.
†*Ibid.*, p. 150.

wendung der Unendlichen Reihen", states that Euler, in all seriousness, set down the astonishing expression

$$1-2+4-8+\cdots=1/3,$$

"und vieles andere mehr", which are quite absurd unless the meaning of the equality sign be redefined.

A single exception to the trend of the times is to be found in the pious Bishop Berkeley. Publishing "The Analyst—A Discourse addressed to an Infidel Mathematician, wherein it is examined whether Object, Principles, and Inferences of the Modern Analysis are more distinctly conceived or more evidently deduced than Religious Mysteries and Points of Faith",* he inveighs against the loose terminologies then prevalent among users of the calculus. He objects to the doctrine of infinitesimals, or fluxions, as they were then called in England, on grounds both technical and empirical. In a fashion reminiscent of Zeno he argues that the notion of ratios existing between absolutely evanescent terms, the "ghosts of departed quantities", is absurd and inintelligible. He invokes Locke to show that the practice of dealing with infinity as a number is equally absurd. Berkeley was right on both counts, and his efforts bore later fruit in pure mathematics, but his was a voice crying in the wilderness. For the most part, the philosophers ignored the mathematicians, and the mathematicians reciprocated.

The latter were not always the losers. Locke, in his "Essay on Human Understanding" treats number in essentially the same manner as did the Greeks, apparently in ignorance of the developments current even in his day. It remained for Leibniz' commentary on Locke, the "Nouveaux Essais"† to point out that extension could with validity be thought of in terms of the number continuum, a fact which Locke had denied.

As the century waned, and as the more immediate results of the calculus were gleaned in, the unsatisfactory state of affairs became apparent. In his later years Lagrange feared that mathematics as a field for research was pretty well played out, and his advice to ambitious young scholars was to turn to chemistry or physics. Of course he was wrong—mathematics was in its infancy, but it remained for another century to disclose this fact.

4. *The Nineteenth Century.* Perhaps the most amazing aspect of the mathematics of the last century has been its continued acceleration of growth. It is no exaggeration to state that the volume of

*Berkeley Works, III.
†V. I., p. 271.

mathematical research published since the beginning of the nineteenth century exceeds the total mathematical output of all previous centuries put together. The nature of this growth has been, one may say, both horizontal and vertical—horizontal in the sense that whole new fields have been discovered and developed, and every existing branch of the subject broadened by new results; vertical in that unifying and simplifying processes of great generality have been applied to the subject as a whole, while rigorous examination has strengthened and deepened the underlying foundations on which the mathematical structure rests. In particular have the latter trends been the result of interplay between the current mathematical and philosophical thought of the age, and a consideration of some of these may be of interest.

Discussion as to the nature of space had taken place since Hellenic times. It will be recalled that Kant, himself a professional mathematician, held that space is not part of the "stuff of experience", but rather an *a priori* form of intuition, inherent in the human mind. Furthermore, what Kant meant by "space" was what we would call euclidean 3-space, and although one might, according to Kant, postulate the existence of beings whose minds intuited space in some other fashion, theirs would not be human minds. Kant's doctrine was adequate for his Newtonian mathematics, but it came to be superseded by the view that space, any space, is a concept, and a concept derived from experience. The revision has come about as the result of two lines of investigation, the one almost as old as geometry itself, the other quite modern. In the first place, it was finally shown that the set of postulates of Euclid did not give the only possible consistent interpretation of space. Many centuries of efforts to prove the postulate of parallels, efforts ranging in time and space from medieval Iran to Renaissance England, culminated in the eventual realization that the postulate could not be proved, and a number of equally valid geometries were developed as a result. In addition to this, later observers, the positivist Mach and others, have shown that the supposed "intuitive space" of Kant is in reality an unconscious comcomposite of two or more experiential "spaces"; visual space, tactual space, and even the space of hearing—audio space, to coin a term. None of these spaces conform to the euclidean metric system. In the space of sight, parallel lines visibily intersect at a point, to prove which one need only look at a straight stretch of railroad track. Tactual space, the space of feeling, is the analogue of a two-dimensional, finite, unbounded, and closed Riemann manifold,* and so on. The

*Mách "Space and Geometry" p. 9.

upshot of it all is that Kant's position is hardly tenable any longer. Neither the old philosopher of Königsberg nor his system are to be condemned for this, however. He himself said:* "Anschauungen ohne Begriffe sind blind", and it is to his imperishable merit to have called attention to the fact that if sensations of space are independent of the qualities of the stimuli which produce them, then predications may be made independently of physical experience.†

Turning now to the romantic school which flourished and had great influence in the nineteenth century, especially in literature, it is at once to be suspected that romanticism and mathematics have but little in common, for your romantic will have no truck with reason, while the mathematician can hardly get along without it. It is true indeed that Napoleon, the greatest romantic of them all, was most intimate with Monge, the inventor of descriptive geometry, and Fourier, who applied analysis to the study of the conduction of heat. Both of these latter were first-rate mathematicians, yet if the episode proves anything at all it shows only that eminence in mathematics is no necessary indication of a sense of discrimination in anything else.

But in the intensive preoccupation of the romantics with the ego, in their efforts to explain the workings of the self, may be found the motivation for the work of Sylvester, Poincarè, and others in beginning what one may call the psychology of mathematical research. In this connection, and in contrast to the insistence on logical rigor which was a contemporary trend, it is well to recall Poincarè's doctrine that usually genius is led to new discoveries by a bold stroke of intuition, rather that by consciously logical thought-processes. He does not deprecate the essential part played by logic in verifying results, but he does hold that logic does not give the results to be tested. A completed theorem, says he, in the symmetry and self-sufficiency of its final logical form, is like a masonry arch, and just as most people, in comtemplating the arch, forget the supporting falsework and scaffolding whereby it was created, so do most of us ignore the falsework and scaffolding of mathematics, namely intuition.

At first blush it would seem that the idealists, with Hegel at their head, would have had little more influence than the romanticists, for Hegel himself was innocent of mathematics, and the problems he considered had no direct contact with it. Closer consideration, however, brings out relations between many of the tenets of idealism and those of the mathematics of the nineteenth century. Hegel was above all things an organizer—his was a System of systems, and the mathe-

*"Kritik der reinen Vernunft", p. 75.
†Mách, l. c., p. 34.

matical counterpart to his systematization was the tendency toward unification and generalization noted in the opening paragraph of the section. It was during this century that Dedekind and Georg Cantor gave the first satisfactory treatment of transcendental and irrational numbers. Their use of the correspondence between the points of a line and the elements of the number continuum made possible the "arithmetization of geometry", the reduction of geometry to analysis. The introduction of the abstract spaces of Riemann, Hilbert, Finsler, and others has so furthered this process that it is now next to impossible to tell where analysis begins and geometry ends. Thus did the long sundered couple, Analysis and Geometry, whose nuptials had been solemnized by Pythagoras himself, but who had been all too soon estranged from each other by the paradoxes of Zeno, find final reconciliation in the nineteenth century, whence every indication is that the union has been a fruitful one.

A second, and most important effort of the century was expended into taking the mysticism out of mathematics. Again this was in accord with the current dictum that truth is synonymous with consistency. As has been seen, the realization that any mathematical discipline is a body of deductions from a set of abstract axioms was anything but new, but the realization that the basic axioms were assumptions rather than "self-evident truths" was new, and often most disturbing. In the course of these ages, these axioms had come to be invested with a legendary absolutism. To deny any one of them was regarded almost as a moral transgression. But the new views were driven forcibly home by technical developments in mathematics itself. The exhibition of hypercomplex number systems, notably quaternions, in which x times y is not equal to y times x, is a case in point. After consistent systems of non-euclidean geometry had been brought forward, it was easier to see that the question of whether or not euclidean geometry is "true" is wholly without meaning.

Other schools and other men were not without influence. Comte, like Kant a professor of mathematics, together with the other positivists named above was among these. In the United States the voice of the pragmatist was heard in the land. The much-quoted statement of Kroneker, with which we close, smacks strongly of the rising tide of twentieth century pragmatism. Said he: "Die ganzen Zahlen hat der liebe Gott gemacht; alles andere in der Mathematik ist Menschenwerke."

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Teaching Centroids and Moments of Inertia Simultaneously

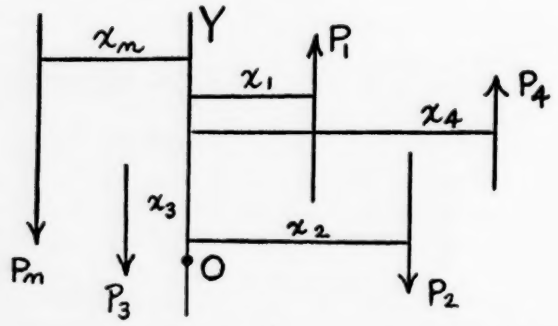
By H. S. KALTENBORN
Louisiana Polytechnic Institute

Two applications of the process of summation that are frequently given in integral calculus are the determination of the centroid and the moment of inertia of an area or volume. In many cases, the student meets these concepts of centroid and moment of inertia for the first time in the course in calculus. He is apt to find these topics difficult, and he may acquire only a hazy understanding of them. He may even obtain the false impression that the radius of gyration for an area or volume with respect to a line of reference is the same as the distance of the centroid from that line.

The writer has employed rather effectively the method described in this paper for teaching centroids and moments of inertia simultaneously. The centroid of an area or volume is found from the first moment of the area or volume, and the moment of inertia is the second moment. By determining the centroid and moment of inertia from moments, the formal procedure is simplified for the student since the method is essentially the same for both. Moreover, as the difference between the formulas for finding the centroid and the moment of inertia is clearly brought out, there is much less danger of the student confusing the two.

Only slight attention is paid in this paper to the physical meaning of the centroid and moment of inertia of an area or volume. While it would be desirable to discuss this question if there is sufficient class-time to do so, the writer feels that the main problem of the calculus class is to learn how to compute the position of the centroid and the value of the moment of inertia for a given area or volume by means of integration. It should be sufficient to state that a knowledge of the position of the centroid and the value of the moment of inertia is of great importance in some problems in mechanics and engineering.

As an introduction to the problem of finding the centroid and moment of inertia of a plane area, consider the *definitions* of the moments of a system of parallel forces about a point. Let the forces be $P_1, P_2, \cdots, P_n$, acting at distances $x_1, x_2, \cdots, x_n$ from the point of reference O, or what is the same thing, from a line of reference through O and parallel to the forces. Introduce a sign convention for both force



and distance, say force positive if directed downwards and distance positive if to the right of the line of reference OY. Then place

$$F = \left\{ \begin{array}{l} \text{first moment of the system of} \\ \text{forces with respect to } OY \end{array} \right\} = \sum_{i=1}^{n} x_i P_i$$

$$S = \text{second moment of the system} = \sum_{i=1}^{n} x_i^2 P_i$$

$$T = \text{third moment of the system} = \sum_{i=1}^{n} x_i^3 P_i$$

. . . . . .

$$K = k\text{th moment of the system} = \sum_{i=1}^{n} x_i^k P_i.$$

These moments $F, S, T, \cdots, K$ are simply numbers, of course, associated with the given forces and their distances from the line of reference. In practice only the first moment is ordinarily used, and F is called *the* moment of the forces about O (or, as we consider it here, the moment with respect to OY).

Now suppose it is desired to express each of these numbers $F, S, \cdots, K$ in the form of a single product, as a force times a suitable power of a distance. Let the single force be the resultant of the original set; viz.,

$$R = \sum_{i=1}^{n} P_i.$$

Determine a set of distances $d_1, d_2, \cdots, d_k$ so that

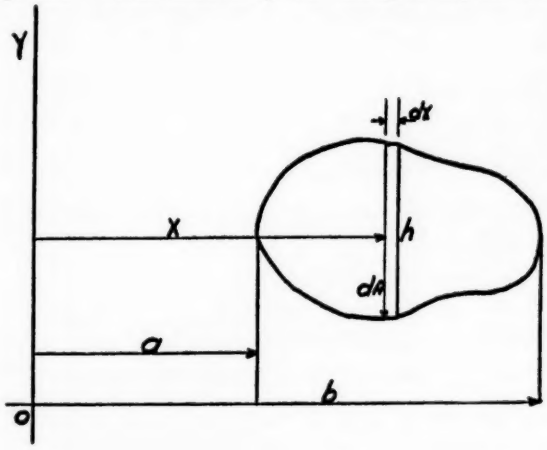
$$F = R \cdot d_1, \quad S = R \cdot d_2^2, \quad T = R \cdot d_3^3, \cdots, K = R \cdot d_k^k.$$

The distance d_1 defined by the first of these equations is simply a distance associated with the given forces and the line of reference,

such that the product of the resultant R and d_1 is the same as the sum of the products of the separate forces P_i and their corresponding distances x_i. Likewise, the distance d_2 is such that the product of R and the *square* of d_2 is the same as the sum of the products of P_i and the *square* of the corresponding distances x_i; and similarly for $d_3, \cdots, d_k$.

In general, these distances $d_1, d_2, \cdots, d_k$ are all different. In particular, d_1 is different from d_2! If the forces are all positive, then every term of S is positive, but F consists of both positive and negative terms, depending on whether the corresponding distances x_i are positive or negative. The addition of another force to the set might increase or decrease F, but it would always *increase* S.

Now consider the moments of a plane area about a line in its plane. This area may be thought of, in the customary manner, as composed of narrow strips parallel to the line of reference OY. Treat each strip dA as a force acting downwards at a distance x from OY, and compute the moments of the area about OY, using integration instead of finite summation. As is the case of the forces P_i above, place



$$F = \left\{ \begin{array}{l} \text{first moment of total area } A \\ \text{with respect to } OY \end{array} \right\} = \int x\, dA = \int_a^b x\, h\, dx$$

$$S = \text{second moment of total area } A = \int x^2 dA = \int_a^b x^2 h\, dx$$

$$T = \text{third moment of total area } A = \int x^3 dA = \int_a^b x^3 h\, dx$$

. . . . . .

$$K = k\text{th moment of total area } A = \int x^k dA = \int_a^b x^k h\, dx.$$

Each of these moments $F, S, \cdots, K$ is simply a *number* associated with the given area A and the line of reference OY. In practice only the first two moments are ordinarily used.

The total area A is the resultant of the separate areas dA; i. e.,

$$A = \int dA = \int_a^b h\,dx.$$

The second moment S is usually denoted by the letter I, and is called the *moment of inertia* of the area A with respect to the line OY. Thus

$$I = \int x^2 dA$$

is simply a number associated with the area and the line of reference. It is a quantity that appears in practical problems in mechanics and engineering, and it is so important that it has been given the special name *moment of inertia.*

As in the case of the first two moments for the forces P_i, determine distances $d_1 = \bar{x}$ and $d_2 = r$, so that

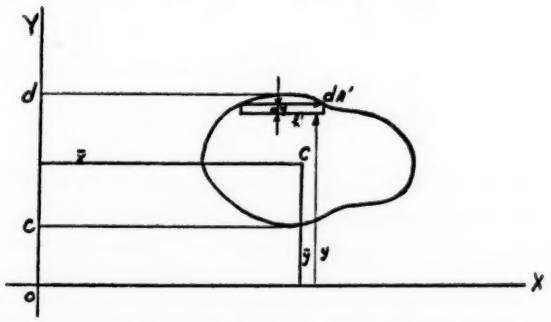
$$F = A \cdot \bar{x} = \int x\,dA, \qquad I = A \cdot r^2 = \int x^2 dA.$$

The distance r defined by the second of these equations is simply a distance associated with the area A and the line of reference OY. It is regarded as of sufficient importance in practical problems that it, too, has been given a special name: *radius of gyration* of the area A with respect to the line OY. The radius of gyration is thus the distance which would yield the same value for the moment of inertia (or second moment) of the entire area A if this area were all located at this distance r from the line of reference.

The distance $\bar{x}$ defined by the first of the equations above is the distance from the line of reference OY to a special point C associated with the area A. It is the distance which would produce the same value for the first moment of the entire area A if this area were all located at this distance $\bar{x}$ from the line of reference. The distance $\bar{x}$ is *not* the same as the radius of gyration.

The special point C can be determined completely by finding its distance $\bar{y}$ from a second line of reference OX, perpendicular to the first, as shown in the figure. The distance $\bar{y}$ is such that

$$A \cdot \bar{y} = \int y \, dA' = \int_c^d y \, h' dy,$$



where dA' is a strip of A parallel to OX. The point C is called the *centroid* of the area A. Thus $\bar{x}$ and $\bar{y}$ are the distances from the lines of reference to the special point C associated with the area A.

After the theory has been developed, a numerical example should be given, of course, to show how the centroid of a plane area and its moment of inertia about a given line are computed. The class is then in a position to handle either or both types of problems.

The procedure for locating the centroid of a volume and finding the value of the moment of inertia of the volume with respect to a plane may be developed in a similar manner by using a plane of reference and taking the first and second moments of the volume with respect to the plane.

The determination of the centroid and moment of inertia of a volume seems to be easier for the student when these topics are treated simultaneously than when they are treated separately as is customary. The simultaneous treatment of centroids and moments of inertia has the further advantage that less class-time is required for the completion of these two applications of integration.

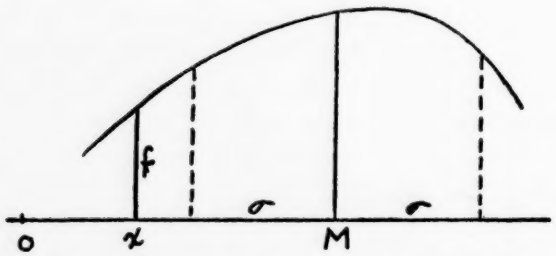
The teaching of centroids and moments of inertia by the method of moments presents a good opportunity to point out to a class how the same theory of moments is used in statistics. If x denotes the variate and f the frequency, the moments are, by definition,

$$V_1 = \text{first moment} \quad = \Sigma x \cdot f$$

$$V_2 = \text{second moment} = \Sigma x^2 \cdot f$$

$$V_3 = \text{third moment} \quad = \Sigma x^3 \cdot f$$

$$V_4 = \text{fourth moment} = \Sigma x^4 \cdot f.$$



The *mean* M, or arithmetic average, of the distribution is found by dividing the first moment by the total frequency $N=\Sigma f$; that is,

$$M=\frac{\Sigma x\cdot f}{N} \quad \text{or} \quad M\cdot N=V_1.$$

The relation $M\cdot N=V_1$ is comparable to the relation $F=R\cdot d_1$ for the system of forces considered above.

The *standard deviation* σ of the distribution may be found from the second moment V_2 by means of the formula

$$\sigma^2=\frac{V_2}{N}-M^2 \quad \text{or} \quad N\cdot\sigma^2=V_2-M^2\cdot N.$$

A simpler formula for σ^2, comparable to the relation $R\cdot d_2^2=S$ for the forces, may be obtained by taking the moments about the mean. The standard deviation is usually defined and computed by means of such a formula.

The mean and the standard deviation are of fundamental importance in the discussion of a frequency distribution. Two other quantities which are frequently employed in such a discussion are Skewness and Kurtosis. For computing the values of these quantities, the third and fourth moments V_3 and V_4 (or corresponding moments about the mean) are required.

Problem Department

Edited by
ROBERT C. YATES and EMORY P. STARKE

This department solicits the proposal and solution of problems by its readers, whether subscribers or not. Problems leading to new results and opening new fields of interest are especially desired and, naturally, will be given preference over those to be found in ordinary textbooks. The contributor is asked to supply with his proposals any information that will assist the editors. It is desirable that manuscript be typewritten with double spacing. Send all communications to ROBERT C. YATES, Mathematics, L. S. U., Baton Rouge, Louisiana.

SOLUTIONS

No. 425. Proposed by *D. L. MacKay*, Evander Childs High School, New York.

Given circles $O(R)$ and $O'(r)$, A a fixed point on circle O, B a fixed point on circle O'. Construct parallel chords AC and BD in circles O and O' respectively so that $(AC)(BD)$ is a maximum. A solution without the use of the Calculus is desired.

Solution by the *Proposer*.

Let AB cut circle O' again in E. Set $\angle CAB = \angle DBE = x$, $\angle OAB = \alpha$, $\angle O'BE = \beta$. Then $(AC)(BD) = 4R \cdot r \cdot \cos(x-\alpha) \cdot \cos(x-\beta)$ is a maximum when

$$\cos(x-\alpha) \cdot \cos(x-\beta) = (1/2)[\cos(2x-\alpha-\beta) + \cos(\beta-\alpha)]$$

is a maximum, i. e., when $\cos(2x-\alpha-\beta)$ is a maximum since α and β both constants. Hence

$$2x-\alpha-\beta=0 \quad \text{and} \quad x=(\alpha+\beta)/2.$$

No. 432. Proposed by *Paul D. Thomas*, Lucedale, Miss.

Three concurrent lines through the vertices of triangle ABC meet the respective sides BC, AC, AB in the points A', B', C'. Circles are described on the segments AA', BB', CC' as diameters. The radical axis of the circle on AA' with the circumcircle of ABC meets BC in P_1. Similarly, using the other two circles with the circumcircle, points

P_2 and P_3 are located on AC and AB. Prove that P_1, P_2, P_3 are collinear. A synthetic proof is desired.

Solution by *Walter B. Clarke*, San Jose, California.

Let A_h, B_h, C_h be the feet of the altitudes of ABC and let A_r, B_r, C_r be the points where the circles with diameters AA', BB', CC', respectively, meet the circumcircle again.

Since CC_r is their radical axis, it is evident from the secant property, first of the circle with diameter CC', then of the circumcircle, that

$$(P_3C_r)(P_3C) = (P_3C')(P_3C_h) = (P_3B)(P_3A).$$

Using this, it may be shown after some reduction that

$$\frac{(AP_3)}{(P_3B)} = \frac{(AC')(AC_h)}{(C'B)(C_hB)}; \quad \frac{(BP_1)}{(P_1C)} = \frac{(BA')(BA_h)}{(A'C)(A_hC)};$$

$$\frac{(CP_2)}{(P_2A)} = \frac{(CB')(CB_h)}{(B'A)(B_hA)}.$$

The product of these three quantities is unity, for

$$(AC_h)(BA_h)(CB_h)/(C_hB)(A_hC)(B_hA) = 1,$$

since the altitudes are concurrent; and

$$(AC')(BA')(CB')/(C'B)(A'C)(B'A) = 1,$$

since the cevians are concurrent. Accordingly, by the theorem of Menelaus, P_1, P_2, P_3 are collinear.

Also solved by the *Proposer*.

No. 433. Proposed by *D. L. MacKay*, Evander Childs High School, New York.

Given circle (O), tangents PA and PB, point C on OA such that $BC = PB$, and chord BD bisecting angle PBC. Prove that the projection of BD on diameter AOE equals the radius of circle (O).

Solution by *Walter B. Clarke*, San Jose, California.

Let G and H be the feet of the perpendiculars dropped from B to PA and from D to BG, respectively. Then HB is the projection of DB on AE. Since angle DBP, equal to angle CBD, is measured by half the arc DB, then angles CBP and DOB are equal. Accordingly, from similar triangles,

$$DB/PC = OB/PB.$$

Moreover, triangles DBH and CPA are similar and thus

$$DB/PC = HB/PA = OB/PB.$$

But, since $PB = PA$, this last equation yields $HB = OB$, the radius of circle (O).

No. 434. Proposed by *Paul D. Thomas*, Lucedale, Miss.

Given $$\lim_{x \to m} \log \sum_{n=0}^{\infty} a/x^n = 0.$$

Determine the relation between a and m.

Solution by *Arnold Kramish*, student, University of Denver.

$$\lim_{x \to m} \log \sum_{n=0}^{\infty} a/x^n = \log \lim_{x \to m} \sum_{n=0}^{\infty} a/x^n = \log \lim_{x \to m} \lim_{k \to \infty} \sum_{n=0}^{k} a/x^n$$

$$= \log \lim_{k \to \infty} \lim_{x \to m} \sum_{n=0}^{k} a/x^n = \log \lim_{k \to \infty} \sum_{n=0}^{k} a/m^n = \log \frac{am}{m-1},$$

where the change of order in the double limit is justified provided the series converges uniformly in some interval containing m, which is the case when $|m| > 1$. Finally the result will be zero for $am/(m-1) = 1$, whence we have

$$|m| > 1, \quad a = 1 - 1/m.$$

Also solved by *C. M. Ablow*, *A. B. Farnell*, and the *Proposer*.

No. 436. Proposed by *W. V. Parker*, Louisiana State University.

Find the equation of the curve tangent to the circle $x^2 + y^2 = a^2$ at (O,a) such that its tangent at (h,k) is parallel to the tangent to the circle from (O,k).

Solution by *Paul D. Thomas*, Lucedale, Miss.

The polar of (O,k) with respect to the given circle is $y = a^2/k$, which meets the circle in $\pm (a/k)\sqrt{(k^2 - a^2)}$. The slopes of the tangents to the circle at these points are

$$\pm (1/a)\sqrt{(k^2 - a^2)}.$$

Accordingly, the slope of the tangent to the required curve at (h,k) is

$$dk/dh = \pm (1/a)\sqrt{(k^2 - a^2)}.$$

Integrating, $$\cosh^{-1}(k/a) = h/a + C.$$

Since the curve must pass through $(0,a)$, then $C=0$ and, reverting to x,y notation, its equation is

$$y=a\cdot\cosh(x/a),$$

a *catenary* with vertex at $(0,a)$.

Also solved by *J. M. Hurt, L. Shenfil,* and the *Proposer.*

Editor's Note: This noteworthy and simple construction for a tangent to the catenary should be better known. Since the technique necessary here is generally acquired by the middle of a course in elementary calculus, this problem is well within the range of the average sophomore. References to texts in which the problem appears would be welcome.

No. 439. Proposed by *Walter B. Clarke,* San Jose, California.

Given angle C. Construct a triangle ABC in which $b-a=(1/2)h_c$.

Solution by *D. L. MacKay,* Evander Childs High School, N. Y.

Let it be required that $b-a=(k/l)h_c$, $b>a$. The specie of triangle ABC is fixed, for, since

$$h_c=c\cdot\sin A\cdot\sin B/\sin C \text{ and } b-a=c(\sin B-\sin A)/\sin C, \text{ then}$$

$$(\sin B-\sin A)/\sin A\sin B=\csc A-\csc B=k/l. \tag{1}$$

Angles A and B are determined by (1) and the further relation $A+B=\pi-C$.

For an analysis, suppose the construction has been made. Then, prolong AC to E so that $CE=CB$ and from D, the midpoint of arc ACB of circumcircle $O(R)$, draw diameter DH cutting AB at G.

From the isosceles triangle CEB, angle $CEB=C/2$ and A, B, E lie on the circle $D(DA)$. Then

$$(2R)\cdot h_c=a\cdot b=(CA)(CE)=(DA)^2-(DC)^2. \tag{2}$$

If DF is perpendicular to AE at F, then $AF=(AE)/2=(a+b)/2$ and $CF=(b-a)/2$. Also since $\angle DAB=(\pi-C)/2=\angle DCA$, the similar triangles ADG and DCF give

$$CF=(AG)(DC)/(AD). \tag{3}$$

Dividing (3) by (2),

$$(CF)/(2R)h_c=(AG)(DC)/(DA)(DA^2-DC^2),$$

whence, substituting k/l for $(b-a)/h_c$, we have:

$$(DA)^2/(DC)-(DC)=2R\cdot l\cdot(AB)/k\cdot(DA). \tag{4}$$

With the selection of any segment for AB, the right member of (4) can be reduced to a segment m and we have to construct two lines $x=(DA)^2/(DC)$ and $y=DC$, given their product $(DA)^2$ and their difference m.

Hence, circumscribe circle O about isosceles triangle ADB in which we know AB and $\angle ADB=\angle C$. Circle $D(y)$ cuts this circle in C, the third vertex.

Closely connected with this problem are the constructions of triangles ABC given

(1) c, C, h_c+b-a

(2) $c, C, (a+b)/h_c$

(3) $c, C, b-a-h_c$.

Also solved by *L. Shenfil.*

PROPOSALS

No. 458. Proposed by *J. M. Hurt*, Sanatorium, Texas.

What should be the ratio of braking power between the front and rear wheels of an auto to produce maximum braking power?*

No. 459. Proposed by *V. Thébault*, Tennie, Sarthe, France.

In every system of enumeration with base $B>2$, where B^2+1 is prime or is the product of prime numbers to the first power, there are no other eight-digit squares of the form *ababcdcd* than those whose root is a four-digit number *mnmn*, and there are always at least two of these. What is the statement for the other values of B^2+1?

No. 460. Proposed by *William N. Huff*, The Hill School, Pottstown, Pennsylvania.

On each edge of a regular octahedron choose one point such that the two opposite edges are divided in the ratio a/b and the other edges in the ratio b/a. Find a/b so that these points are vertices of a regular icosahedron.

No. 461. Proposed by *C. E. Springer*, University of Oklahoma.

Consider a simplex determined by $n+1$ points on a central hyperquadric in n-space, together with the simplex determined by the $n+1$

*On the 1942 model Ford the ratio of front to rear wheel braking power is 3/2.

tangent hyperplanes at these points. Show that the center of the hyperquadric and the centroids of the simplexes either all coincide or are all distinct. (This is an extension of a problem which appeared in the American Mathematical Monthly, 1936, p. 433.)

No. 462. Proposed by *Paul D. Thomas*, Lucedale, Miss.

A variable line turns about the orthocenter of a fixed triangle. Find the locus of the orthopole of this line with respect to the triangle.

No. 463. Proposed by *E. P. Starke*, Rutgers University.

Prove the construction for the tangent to any conic at one of its points P : join P to a focus F; let the perpendicular to PF at F meet the corresponding directrix at D; DP is the required tangent.

No. 464. Proposed by *D. L. MacKay*, Evander Childs High School, New York.

Show that for any right triangle ABC the sum of the distances of the orthocenter H from the vertices equals the sum of the diameters of the inscribed and circumscribed circles.

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Bibliography and Reviews

Edited by
H. A. SIMMONS and JOHN W. CELL

A Mathematician's Apology. By G. H. Hardy. Cambridge: at the University Press. New York: the Macmillan Company, 1940. vii+93 pages. Price $1.00.

Although a *Mathematician's Apology* is addressed to non-mathematicians, it should appeal to any thoughtful man, mathematician or not. Its purpose is to answer the questions, "Why is it really worth while to study mathematics? What is the proper justification of a mathematician's life?" The answer to such questions can be given only by a man who is worthy of the name of "mathematician" and who is willing to give a frank appraisal of his own life. Professor Hardy can speak with authority and he has spoken frankly.

To justify a man's choice of mathematics as a career, Professor Hardy shows that such a choice offers an exceptional opportunity to gratify ambition, the driving force behind nearly all the best work of the world. He is speaking of men of ability and dismisses the problems of the ordinary man as unimportant.

To show the value of mathematics, Professor Hardy discusses the permanence of the ideas from which mathematical patterns are formed. The appeal of "serious" mathematics is compared with the appeal of the genuine but "trivial" mathematics of chess. Examples of "real" mathematical theorems are given to show the qualities of generality and depth which characterize "serious" mathematics.

A real mathematician, and Hardy is one, is interested in mathematics as a creative art, but no discussion of the worthwhileness of mathematics would be complete without considering its utility. This requires a statement of the distinction between pure and applied mathematics, between physical reality and the reality of ideas. Hardy distinguishes two mathematics, that which is useful, comprising the large bulk of elementary mathematics, and that which he calls "real" mathematics, most of which cannot be justified on the basis of its utility at all but rather as art is justified.

This is a small volume but a very compact one. It is a challenge to every reader to examine the justification of his own life as frankly as Hardy has done, as well as a personal and direct answer to the controversial questions which the author has proposed for himself in the beginning.

A review of this particular book would not be complete without mention of one of its most outstanding characteristics. It is written in a perfect essay form and in a very simple, direct, and pleasing style.

Wright Junior College, Chicago. VIRGINIA MODESITT.

The Teaching of Secondary Mathematics. By Charles H. Butler and F. Lynwood Wren. McGraw-Hill Book Company, Inc., New York, 1941. xii+514 pages.

Books on the teaching of mathematics are not as numerous as similar books in other subject matter areas; hence, this new publication treating the technics and problems of the mathematics teacher is most welcome. The present volume is the contribu-

tion of two authors whose practical experiences both as teachers of mathematics and teachers of methods ably qualify them for the task they undertook and successfully consummated.

Since "those persons who concern themselves with the teaching of mathematics in secondary schools usually find their interests springing mainly from three sources," the book has been divided into three parts. Part I (62 pages), "The Place and Function of Mathematics in Secondary Education," is intended for the student of education—administrator, supervisor, or classroom teacher—who wishes to secure a proper orientation of mathematics into the school system. Part II (178 pages), "The Improvement and Evaluation of Instruction in Secondary Mathematics," is directed toward the administrator and supervisor who wish to improve instruction. Part III (257 pages), "The Teaching of the Special Subject Matter of Secondary Mathematics," is written for the classroom teacher, whose task it is to deal with subject matter and its accompanying instructional problems. The significance of this brief outline is self-evident; its development is this excellent book.

A brief history of the American school from the Latin Grammar School through the present junior college is found in Chapter I. A picture of the evolving program of secondary mathematics, with special reference to the influence of the Committee of Ten, the National Committee of Fifteen, and many others is given in Chapter II. The functional role of mathematics in the secondary school curriculum is presented in Chapter III.

Part II (Chapters IV-X) is devoted to curricular consideration of the materials of instruction, instruments as aids to teaching, interest in mathematics, means to effective instruction, testing, professional preparation of teachers, and supervision of instruction. Of these chapters the one of greatest value is probably the one on means to effective instruction which covers the three fundamental problems of teaching mathematics: teaching for understanding, teaching for assimilation, and teaching for permanence.

Methods of teaching are discussed in detail in the last eight chapters. Here is found a wealth of material on the teaching of arithmetic, algebra, plane geometry, solid geometry, trigonometry, and calculus. Such topics as approximate computation, graphs, fractions, fractional equations, verbal problems, proof in demonstrative geometry, interpolation, and logarithms are analyzed clearly and expertly.

If the authors have erred, it is in attempting too much in a single volume. They have covered the entire field of secondary mathematics from grade seven of junior high school through grade fourteen of junior college, and they have set it in a meaningful perspective of history and philosophy. As a result, they have been forced to generalize in cases where to particularize would seem to be more desirable; they have had to exclude in instances where to include would seem to be better. However, these shortcomings will stimulate the use of the excellent bibliography which is given at the end of each chapter; each bibliography is in two parts, Basic Bibliography and Supplementary Bibliography.

The very good lists of questions at the end of each chapter suggest the suitability of this book as a text for teacher-training classes where it should be highly successful. It is a book which every teacher of mathematics should read and then place in his professional library to be read again and again.

University High School, University of Illinois. MILES C. HARTLEY.

A Study of the Problems in Teaching the Slide Rule. (Contributions to Education, No. 805). By Carl N. Shuster, Ph.D., Bureau of Publications, Teachers' College, Columbia University, New York, 1940. 103 pp. Cloth. $1.85.

That this book merits inclusion among the Contributions to Education (either with capitals or without) is the opinion of this reviewer. It is difficult to agree however, that the title represents precisely the subject matter of the book. Since considerable space is given to wholesome discussion of such topics as accuracy of measurement, approximate computation, and "new" methods for locating the decimal point, the reviewer modestly proposes that the title might well be altered to "A Study of Problems (Not Exclusively) Connected with Teaching the Slide Rule". Especially the editors of textbooks, teachers and students of classes in the teaching of arithmetic, and to all who need prodding to cause them to conform to consistent practice in approximate calculation should this little volume be useful. The author states his credo:

> "No textbook should give, and no teacher should allow more significant figures in a final answer in multiplication in division, or in square root than there are in the least accurate item used, and all texts should show by the data used the accuracy desired by the answer. All data in a given problem should be consistent."

These stout words terminate the first portion of the book, which sketches the histories of mechanical computation and decimal fractions and gives brief but clear treatment of the principles and rules of approximate computation. Thus the author points a way toward a discussion of accuracy of measurement and the adequacy of the slide rule as an instrument for computation and gives tables and examples of official legal tolerances and relative errors in the determination of chemical constituents, and closes with the statement that an examination of 3479 problems revealed that over 70% of them could be solved on a ten inch slide rule; over 95% of them, on a twenty-inch rule; and all of them, on a Thatcher or Fuller rule.

The last five chapters of the book relate the author's experiences and conclusions in discovering the variety and relative frequency of errors in slide rule computation, and the effectiveness of methods for the prevention of such errors. Tests, tables, summaries of test results, graphs, etc., all conforming to the statistical etiquette of schools of education, entitle the publishers to the claim that "This is the first experimental study in English on the teaching of the slide rule."

Central Y. M. C. A. College, Chicago, Illinois. JAMES E. DAVIS.

Elementary Mathematical Concepts. (From the Historical and Logical Point of View.) By James H. Zant, Ph.D., and Ainsley H. Diamond, Ph.D. Burgess Publishing Co., Minneapolis, 1941. (Offset reproduction of typewritten copy.) 125 pp. $1.50.

Even in these days of wide departures by textbook writers from traditional content and arrangement, this is an unusual book.

It is unusual to see a book offered for use "for a freshman course for non-science students," and which "could also serve for students in more advanced classes by using the references at the end of each chapter," in which there is not a single equation to be solved.

It is unusual to encounter such a curious melange of more or less fascinating historical items, justification of the fundamental operations on rational numbers such as is found in ninth-grade (and earlier) texts, and a brief but not too coherent intro-

duction to sundry items from the mathematics of finance, all rolled together into one book.

In a college text it is unusual to find, as in this one, that most of the rather few exercises could be worked out by an average ninth-grade pupil, and many of them by grade school pupils.

It is unusual to encounter in a college text such use of English and such lack of clarity of thoughts as one finds in the quotation below, from page 81 of the text:

> "Life insurance is essentially the same thing. A large number of people pay small sums of money each year and those who die during the year have $1000 paid to their dependents or to their estate (*sic*). It is not so simple as this because the likelihood of a person (*sic*) dying depends on many things and increases with his age and *the state of his health.* (Italics mine.) Determining just what each person should pay becomes a somewhat complicated mathematical problem and the results are based on the tables of what has happened in previous years. The table used by the life insurance companies in this country is called The American Experience Table of Mortality and gives the number of people out of a selected group *living and dying each year.* (Italics mine.) This selected group of people are (*sic*) those who have owned life insurance in the past. No one knows whether *the same things* (Italics mine) would be true for the group of people who do not own insurance. We could probably be rather sure that the people who own insurance are as a whole a healthier group, since they have all passed a physical examination at least one time in their life. (*Sic*)... As in the case of fire insurance a small outlay per year will make impossible the loss of a big part of your property; so with life insurance, small yearly payments make it possible to leave a reasonably large estate long before it would be possible to acquire one by small savings." (No mention of early death as a necessary condition on the validity of the preceding sentence.)

To go on with examples of definitions, problems, and concepts deemed objectionable by the reviewer would be to consume space which might well be devoted to the review of another book.

Central Y. M. C. A. College, Chicago, Illinois. JAMES E DAVIS.

The Relative Merits of Three Methods of Subtraction. By John Theodore Johnson, Bureau of Publications, Teachers College, Columbia University, New York City, 1938. 76 pages. $1.60.

Those who teach the fundamental operations with numbers in the elementary school realize the importance of the problem dealt with in Mr. Johnson's study. Five years ago one of the questions which the elementary school teacher was asking the investigator to answer was, "Which is the best method of subtraction?" Mr. Johnson's contribution, which brings together the evidence on the relative merits of three methods of subtraction, has been gratefully received by teachers of arithmetic.

The problem is clearly described in the first chapter together with descriptions and illustrations of the methods of subtraction to be considered.

The historical material of Chapter II furnishes the kind of background which gives teachers breadth of view and confidence in their knowledge of the subject. The quotations from old books are well chosen and add considerably to the reader's pleasure. The quotation from the Treviso Arithmetic shows that the emphasis was on "steps" of

procedure rather than on helping the learner to get the mental satisfaction of seeing that the procedure was mathematically logical and led to correct results. Without the material of Chapter II the teacher of subtraction would not know his subject.

The review of "past investigations" is made objectively. The author's statement that "the Taylor report proves nothing with regard to the efficiency of either method" will be concurred in by many. The criticism that the Buckingham study carried so large a probable error as to greatly diminish the reliability of the results is justified. Mr. Johnson has given sufficient data from each study to give the reader an opportunity to form his own opinion regarding the quoted conclusions by the author of each study. Mr. Johnson is conservative in concluding from the seven experiments, which he credits with some significance, that "all of them show that the decomposition method is inferior to a method that increases the subtrahend—either the equal additions method or the Austrian."

In approaching the report of his own investigation, Mr. Johnson instructs the reader as to the meaning of the scientific method of difference. It is explained that in subtracting in the following exercise,

$$\begin{array}{r} 6234 \\ 3786\,, \\ \hline \end{array}$$

"the subject performs these subtractions:

$$\begin{array}{rrrrr} 14 & 12 & 11 & & 5 \\ 6\,, & 8\,, & 7\,, & \text{and} & 3 \\ \hline \end{array}$$

and, in addition to this, he performs the extra operation of borrowing (so-called), necessary to this method. The extra time it takes for subtracting

$$\begin{array}{r} 6234 \\ 3786 \\ \hline \end{array}$$

over the time it takes for the simple facts

$$\begin{array}{rrrrr} 14 & 12 & 11 & & 5 \\ 6\,, & 8\,, & 7\,, & \text{and} & 3 \\ \hline \end{array}$$

is clearly due to the extra operation or technique of borrowing. As with borrowing so with the other methods, the time consumed by the scheme is the total time for the exercise less the time for the primary facts involved.

By the use of two tests, one composed of exercises necessitating borrowing, equal additions or carrying, and the other test composed of the primary facts used by the particular method, it is possible to establish the time used for operating the subtraction technique.

In Mr. Johnson's experiment six thousand pupil-tests were given by the experimenter personally. With some assistance, the examiner scored all the tests. All groups took the same test. Each pupil was asked to record on a separate sheet of paper just what he said to himself when he carried out these two subtractions,

$$\begin{array}{rr} 82 & 600 \\ 37\,, & 146\,. \\ \hline \end{array}$$

The individual pupils were classified into three groups on the basis of the method shown on these individual papers.

Data are presented showing correlation coefficients and their probable errors establishing higher reliability in time than in accuracy. The author says that this is probably due to the fact that some errors are accidental. The data also show that the groups are highly comparable on the basis of mental age as well as on the basis of errors made in Test I (primary facts) The data show that on the average each member of the decomposition group made 5.73 more errors in the subtraction exercises than in the subtraction facts test; those of the equal additions group, 4.84 more errors, and those of the additive group, 4.92 more errors. Since there is an increase in errors among all the groups and since these increases are not the same, the differences must be due to the methods used.

The data of Table X set forth the following conclusions:

1. The decomposition method causes 18.4% more errors than the equal additions method and (at the same time) requires 15.5% more time.

2. The decomposition method causes 16.4% more errors than the additive method and requires 67% more time.

3. The decomposition method causes 17.7% more errors than the combined equal additions and additive methods and requires 28.6% more time.

4. The equal additions method causes 1.6% less errors than the additive method and requires 44.5% more time.

A secondary experiment made possible by the data was a comparison between the decomposition group and the combined equal additions and additive groups on the basis of time only. This was made by selecting only those pupils from each group who made no errors in the facts test and only a few errors in the "exercises" test. This eliminated the errors variable and left the time variable.

Data are presented showing the equivalence of the groups on five bases: M. A., I. Q., errors in Test I, errors in Test II, time on Test I.

There is no escaping the conclusion, based on the data, that the borrowing method necessitates about 50% more time for the operation of the borrowing technique than the equal additions and Austrian methods necessitate for the operations of their respective techniques.

A final chapter summarizes the conclusions and presents pedagogical implications. Significant statements in this chapter on pedagogical implications are: "The Austrian method or the equal additions method requires less concentration and strain than the decomposition method"; "It is a question whether efficiency in later performance should be sacrified for the sake of an initial stage of more facile explanation"; "To summarize, all the available evidence seems to be definitely in favor of the Austrian method as the most efficient and the most easily taught procedure in subtraction; the equal additions method comes next in order of merit and should be the choice of those who, for some reason, are opposed to the Austrian method. The decomposition method, in view of the evidence now at hand, makes a poor showing, being inferior in both speed and accuracy to either of the other methods."

Northern Illinois State Teachers College. W. B. STORM.

Intermediate Algebra. By H. L. Rietz, A. R. Crathorne, and L. J. Adams. Henry Holt and Company, New York, 1941. viii+248 pages. $1.75.

This book covers a more or less standard type of material that is suggested by the following chapter headings (and associated page numberings): Algebraic Operations (1-18); Linear Equations in One Unknown (19-29); Factoring (30-35); Fractions

(36-54); Functions (55-70); Systems of Linear Equations (71-95); Exponents and Radicals (96-121); Quadratic Equations (122-141); Equations Involving Radicals or Fractional Exponents (142-146); Ratio and Proportion (147-154); Variation (155-162); Progressions (163-174); Mathematical Induction and the Binomial Theorem (175-183); Extension of the Number Concept (184-192); Logarithms (193-209); Systems of Equations Involving Quadratics (210-222). It contains no such modern procedure as fitting a straight line to statistical data, differentiation of a polynomial in x with respect to x, or applying the most elementary parts of the theories of *permutations and combinations* and *probability* to statistical or insurance problems.

For students who have had one year of algebra and who are to take material of the type that this book contains, it should prove to be an excellent text for the reasons which we now proceed to set forth.

The exposition is clear and concise. A teacher is not apt to wish that the authors had begun any discussion in a more elementary or natural way or that they had developed any topic more gradually. Further the exposition that is typical of Rietz and Crathorne as to conciseness and completeness is maintained in this text. When they define a quadratic equation, on page 122, they state that $a \neq 0$ in the equation $ax^2+bx+c=0$; when they solve a system of three linear equations in three unknowns, on page 84, they state that for their solution to be valid it is necessary that the determinant which appears in the denominators of three quotients be not equal to zero; etc. Further, in spite of the fact that the emphasis of the text is naturally on algebraic processes, the book contains 45 attractive figures.

The exercises of the text are thoroughly introduced. The material that precedes any list of exercises is about what any teacher would desire to write to introduce his own class to such material.

The exercises are well graded. It is obvious from a glance through the different lists of exercises that the early ones of any set are simple and that difficulty increases with the number of the problem; and the book contains numerous problems which should, as the authors state, "challenge the best students".

Answers are given to the odd-numbered problems. We solved several problems from the text and compared answers with the authors; there were no disagreements.

The book contains three interesting historical references, which together cover about one page. These references are to Hamilton (Sir William Rowan) for his introduction of an algebra of vectors i,j,k for which the commutative law $ab=ba$ does not hold, to Leibnitz and Cramer for their use of determiants in solving systems of linear equations, and to Diophantus for his celebrated work in connection with indeterminate equations. Since each reference is brief and extremely interesting, both students and teachers are apt to wish that more such references had been given. However, teachers may well regard such references as exemplary and attempt to supply numerous other similar ones for their classes.

Northwestern University. H. A. SIMMONS.

Mathematics for the Aviation Trades. By James Naidich. McGraw-Hill Book Company, New York and London, 1942. x+267 pages.

This book is admittedly elementary in so far as mathematics is concerned for, as the author states in the preface it "has been written for students in trade and technical schools who intend to become aviation mechanics". The reader must merely know how to substitute in formulas, solve simple equations, and "handle" fractions and decimals. The author says further that "this book obviously does not present all the

mathematics required by future aeronautical engineers. All mathematical material which cannot be adequately handled by *elementary arithmetic* is omitted. (The italics are the reviewer's.)

The book is admirably written, well bound, and clearly printed on smooth heavy paper.

Divided into five parts and 16 chapters, the subject matter is presented logically in "jobs", of which there are 106 in the entire book. These jobs "deal with specific aspects of the work that an aviation mechanic may encounter. All the mathematical background required for this work is covered" (also as jobs) in Part I of the book.

The "job" idea is carried consistently to good effect throughout the new material of the book, that is, throughout Parts I to IV; Part V "contains 100 review examples taken from airplane shop blueprints, aircraft-engine instruction booklets, airplane supply catalogs, air craft directories, and other trade literature." For instance, job 1, Chapter 1, is entitled *Learning to Use the Rule;* job 8, Chapter 1, *Division of Fractions;* job 4, Chapter 2, *Division of Decimals;* job 1, Chapter 6, *How to Use a Protractor;* job 1, Chapter 7, *The Bar Graph* (where it is explained how to read a graph based on coordinate axes); job 4, Chapter 8, *The Gross Weight of an Airplane;* job 7, Chapter 8, *Power Loading;* etc. Part III deals with materials, where there are 16 jobs, covering impirical formulas for tensions, compression, and shear, etc. Part IV begins with *Horsepower.* The subject is well presented. The general idea is introduced through the helpful device of pertinent questions with concise answers immediately given. Part IV is rife with clear diagrams and photographs, definitions, formulas, and numerous illustrative examples (a feature successfully employed throughout the book). Many well selected and well graded problems developing the subject of horsepower train the student in the knowledge that he should possess to become an intelligent and effective mechanic. The closing chapters deal with *Fuel and Oil Consumption, Compression Ratio,* and *Valve Timing.* The jobs here cover such questions as *Horsepower-Hours, Gallons and Cost, How Long Can an Airplane Stay Up, Compression Ratio Diagrams,* and *Valve Timing Diagrams.*

The book is carefully written throughout. Each point is clearly explained and great pains have been taken in an immense amount of minute detail.

At the end of the book several pages are devoted to *Tables of Measure* and a recapitulation of the mathematical formulas and the empirical formulas used in the book. There is also an adequate index.

There can be little doubt but that the book is accurate and authoritative. As Mr. Elliot V, Noska, principal of Manhattan High School of Aviation Trades, says in a foreword, "the material in *Mathematics for the Aviation Trades* has been gathered over a period of years. It has been tried out in the classroom and in the shop. For the instructor, it solves the problem of what to teach and how to teach it."

The reviewer agrees with Mr. Noska that "the text therefore fills a real need"... and that "the use of this book will help solve some of the aviation mechanics' problems."

Lake Forest College. H. B. CURTIS.

Brief Trigonometry. By Edward A. Cameron. Reynal & Hitchcock, New York, 1941. $1.25.

This Trigonometry text is designed for a course of about thirty lessons. It contains the essentials of Trigonometry in three chapters with a total of fifty sections or paragraphs. There are over six hundred exercises and more than sixty worked examples. Answers to most of the odd-numbered exercises are given in the back of the book.

The general definitions of the six trigonometric functions of an angle are given first, the specialization to acute angles being made later. Radian measure, inverse functions, and trigonometric equations are introduced fairly early. General methods of solving oblique triangles are left until the last. The tables in the back seem to be the same as Heath's Logarithmic and Trigonometric Tables to Four Places by E. J. Oglesby (Table V and references to it are omitted), and as such are subject to the remarks made by the reviewer in his review of "Trigonometry" by Curtiss and Moulton (see NATIONAL MATHEMATICS MAGAZINE for December, 1940, page 156).

The typography and make-up are good. Instructors teaching "short courses" in Trigonometry will find this text useful for the purpose.

University of Arkansas. EDWIN G. H. COMFORT.
(Kelly Field, Texas)

To Discover Mathematics. By Gaylord M. Merriman. John Wiley & Sons, Inc., New York; Chapman & Hall, Limited, Lndon, 1942. xi+435 pages. $3.00.

The *discovery* takes place in a preface, thirteen chapters, an appendix of very interesting notes and examples, an author index, and a subject index. Chapter headings together with a brief statement of contents may be of value:

1. *The Fountainhead* (discussion of number);
2. *Magic of the Mind* (methods of mathematical proof);
3. *The Divine Integer* (including a bit of number theory);
4. *Work of Man* (fractions, negative numbers, zero, irrational and transcendental and "compound" numbers, the Dedekind cut);
5. *Algebraic Counterpoint* (laws of algebra, binomial theorem, combinations, permutations, probability, discriminant, groups);
6. *Euclid Alone?* (old Greek problems, modern geometries);
7. *Geometry's Royal Road* (analytic geometry);
8. *Declaration of Dependence* (study of functions, elementary theory of equations);
9. *Eventually and Forever After* (limits—sequences, continuity, derivatives, area under a curve, infinite series);
10. *Theme with Variations* (the derivative with extreme values, inflections, and related rates; the integral with areas and volumes);
11. *Power—and Snowballs* (logarithms with compound interest and annuities, the exponential or power function);
12. *Period Piece* (trigonometry, Fourier series, DeMoivre's theorem);
13. *Peak in Darien* (purposes of mathematics, transfinite numbers, the beauty of mathematics).

The tour director on this voyage is not afraid to mix mathematics and a sense of humor. He quotes Gilbert and Sullivan, Felix Klein, Lewis Carroll ("Alice"), and Whitehead quite impartially and very aptly. Definitions given throughout the book are excellent, and what is needed for a definition is very well explained. A program ranging from quadratic equations and the addition of fractions to group theory and transfinite numbers seems rather ambitious for a "required survey course" (for freshmen ?), but certainly produces a volume that will be of interest to young students, to mathematics clubs, and to college mathematics teachers asked by non-mathematical

friends for an explanation in words of one syllable of $(-2)(-3)=6$, the peculiar behavior of zero, the fourth dimension, or non-Euclidean geometries.

Some statements will rouse argument. The author contends that the idea of calculus can be understood without algebra, trigonometry, or analytic geometry; but surely he will admit that the knowledge of all of them will enrich calculus for the exceptional student and make it bearable for the "weak sister". When a student has "loathed his preparatory school memory course in Euclidean geometry", the explanation may be not in the subject but in the fact that his instructor was the French teacher or athletic coach who happened to have free hours at the *right* time. The term "mathematical induction" is a "misnomer" according to the author I wonder.

This book is delightfully quotable. Here are a few random samples. "Imagination... is as important to the mathematician as to the poet." "Mathematics is not a slave to calculation, but is of the essential nature of a creative art." "The limit concept is not armchair fantasy.... It is the stuff of life. Practically every application of mathematics to physics, to mechanics, to engineering, to economics, involves its use, tacit if not direct."

Wellesley College. MARION E. STARK.

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